Journal of Pure and Applied Algebra 214 (2010) 2294–2300

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

On Brown's constant associated with irreducible polynomials over henselian valued fields

Sudesh K. Khanduja *

Department of Mathematics, Panjab University, Chandigarh-160014, India

ARTICLE INFO

Article history:
Received 18 May 2009
Received in revised form 21 January 2010
Available online 21 March 2010
Communicated by R. Parimala

MSC: 12E05; 12J10; 12J25

ABSTRACT

Let v be a henselian valuation of arbitrary rank of a field K and $\tilde{v}$ be the prolongation of v to the algebraic closure $\widetilde{K}$ of K with value group $\widetilde{G}$. In 2008, Ron Brown gave a class $\mathcal{P}$ of monic irreducible polynomials over K such that to each $g(x)$ belonging to $\mathcal{P}$, there corresponds a smallest constant λ_g belonging to $\widetilde{G}$ (referred to as Brown's constant) with the property that whenever $\tilde{v}(g(\beta))$ is more than λ_g with $K(\beta)$ a tamely ramified extension of (K, v), then $K(\beta)$ contains a root of $g(x)$. In this paper, we determine explicitly this constant besides giving an important property of λ_g without assuming that $K(\beta)/K$ is tamely ramified.

1. Introduction

Throughout this paper, v is a henselian valuation of arbitrary rank of a field K and $\tilde{v}$ is the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K with value group $\widetilde{G}$. In 2008, Brown [5], gave a class[1] $\mathcal{P}$ of monic irreducible polynomials over any henselian valued field (K, v) (which coincides with the class of all monic irreducible polynomials when (K, v) is maximally complete) satisfying the following property:

To every $g(x)$ belonging to $\mathcal{P}$, one can associate a constant λ_g belonging to $\widetilde{G}$ such that whenever $K(\beta)$ is a tamely ramified extension of (K, v), β belonging to $\widetilde{K}$ and $\tilde{v}(g(\beta)) > \lambda_g$, then $K(\beta)$ contains a root of the polynomial $g(x)$. Moreover, the constant λ_g is the smallest with the above property. This constant will be referred to as Brown's constant. It will be shown that the condition $\tilde{v}(g(\beta)) > \lambda_g$ is in general weaker than the analogous condition $\tilde{v}(g(\beta)) > 2\tilde{v}(g'(\beta))$ in Hensel's Lemma for guaranteeing the existence of a root of $g(x)$ in a tamely ramified[2] extension $K(\beta)$ of (K, v) (see Corollaries 1.2, 1.5).

In this paper, our aim is to determine explicitly Brown's constant for all possible irreducible polynomials $g(x)$ and to show that this constant satisfies an important property even without the assumption that $K(\beta)/K$ is tamely ramified. We show that this constant can be associated to any monic irreducible polynomial $g(x)$ belonging to $K[x]$ provided $K(\theta)$ is a defectless extension of (K, v) where θ is a root of $g(x)$. Brown's constant will be determined using complete distinguished chains defined below.

A pair (θ, α) of elements of $\widetilde{K}$ is called a distinguished pair (more precisely a (K, v)-distinguished pair) if $[K(\theta) : K] > [K(\alpha) : K]$, $\tilde{v}(\theta - \beta) \leqslant \tilde{v}(\theta - \alpha)$ for every β belonging to $\widetilde{K}$ with $[K(\beta) : K] < [K(\theta) : K]$ and whenever γ belongs to $\widetilde{K}$ with $[K(\gamma) : K] < [K(\alpha) : K]$, then $\tilde{v}(\theta - \gamma) < \tilde{v}(\theta - \alpha)$. Distinguished pairs give rise to distinguished chains in a natural manner. A chain $\theta = \theta_0, \theta_1, \ldots, \theta_s$ of elements of $\widetilde{K}$ will be called a complete distinguished chain for θ with respect to v if

* Tel.: +91 172 2541132.
E-mail address: skhand@pu.ac.in.

1 This class of polynomials arose in a study of the extensions of v to the rational function field $K(x)$ in [4].

2 A finite extension (K', v') of (K, v) (or briefly K'/K) is said to be tamely ramified if (i) it is defectless, i.e., $[K' : K] = ef$, where e, f are respectively the index of ramification and the residual degree of v'/v, (ii) the residue field of v' is a separable extension of the residue field of v and (iii) e is not divisible by the characteristic of the residue field of v.

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2010.02.028

(θ_i, θ_{i+1}) is a (K, v)-distinguished pair for $0 \leqslant i \leqslant s-1$ and $\theta_s \in K$. It is known that a simple extension $K(\theta)$ of (K, v) is defectless if and only if θ has a complete distinguished chain with respect to v (cf. [2, Theorem 1.2]).

For θ belonging to $\widetilde{K} \setminus K$ with $K(\theta)/K$ defectless, we shall denote by $\delta_K(\theta)$ the main invariant associated with θ defined by

$$\delta_K(\theta) = \sup\{\tilde{v}(\theta - \beta) | \beta \in \widetilde{K}, [K(\beta) : K] < [K(\theta) : K]\}.$$

As shown in [2, Theorem 2.4], the above supremum is attained by virtue of the hypothesis that $K(\theta)/K$ is defectless; indeed there exists α belonging to $\widetilde{K}$ such that (θ, α) is a distinguished pair. Let (θ, α) be a distinguished pair with $g(x)$ the minimal polynomial of θ over K. As shown in Lemma 2.2, $\tilde{v}(g(\alpha))$ is independent of the choice of α. Indeed we prove in the following theorem that $\tilde{v}(g(\alpha))$ is Brown's constant associated with $g(x)$ when $K(\theta)/K$ is a defectless extension.

Theorem 1.1. *Let (K, v) be a henselian valued field of arbitrary rank and $(\widetilde{K}, \tilde{v})$ be as above. Let $g(x)$ belonging to $K[x]$ be a monic irreducible polynomial having a root θ with $K(\theta)$ a defectless extension of (K, v). Let α belonging to $\widetilde{K}$ be such that (θ, α) is a (K, v)-distinguished pair. If β is an element of $\widetilde{K}$ with $\tilde{v}(g(\beta)) > \tilde{v}(g(\alpha))$, then there exists a root θ' of $g(x)$ such that $\tilde{v}(\theta' - \beta) > \tilde{v}(\theta - \alpha) = \delta_K(\theta)$. Moreover $\tilde{v}(g(\alpha))$ is the smallest element of $\widetilde{G}$ satisfying the above property.*

The following two results will be quickly deduced from the above theorem.

Corollary 1.2. *Let (K, v), θ, α, β be as in Theorem 1.1. Suppose that $K(\beta)/K$ is a tamely ramified extension. Then $K(\theta)/K$ is tamely ramified and $[K(\theta) : K]$ divides $[K(\beta) : K]$.*

Corollary 1.3. *Let (θ, α) be a (K, v)-distinguished pair and $g(x)$ be as above. Assume that $K(\theta)$ is a tamely ramified extension of K. If β is an element of $\widetilde{K}$ with $\tilde{v}(g(\beta)) > \tilde{v}(g(\alpha))$, then $K(\beta)$ contains a root of $g(x)$.*

The theorem stated below has been proved to conclude that Brown's constant $\tilde{v}(g(\alpha))$ is indeed smaller than $2\tilde{v}(g'(\beta))$, when $g(x)$ has coefficients in the valuation ring of v. This theorem is of independent interest as well.

Theorem 1.4. *Let θ, α, $g(x)$ and β be as in Theorem 1.1. Assume that $K(\theta)/K$ is a tamely ramified extension. Then $\tilde{v}(g'(\beta)) = \tilde{v}(g(\alpha)) - \delta_K(\theta)$.*

The following corollary will be proved using the above theorem.

Corollary 1.5. *Let the hypothesis be as in Theorem 1.4. Assume that $g(x)$ has coefficients in the valuation ring of v. Then $\tilde{v}(g(\alpha)) \leqslant 2\tilde{v}(g'(\beta))$.*

2. Some preliminary results

Let (K, v), $(\widetilde{K}, \tilde{v})$ be as in the preceding section. By the degree of an element α in $\widetilde{K}$, we shall mean the degree of the extension $K(\alpha)/K$ and shall denote it by *deg* α. Recall that a pair (α, δ) belonging to $\widetilde{K} \times \widetilde{G}$ is said to be a minimal pair (more precisely (K, v)-minimal pair) if whenever β belonging to $\widetilde{K}$ satisfies $\tilde{v}(\alpha - \beta) \geqslant \delta$, then *deg* $\beta \geqslant$ *deg* α. It can be easily seen that if (θ, α) is a distinguished pair and $\delta = \tilde{v}(\theta - \alpha)$, then (α, δ) is a minimal pair.

If $f(x)$ is a fixed monic polynomial with coefficients in an integral domain R, then each $g(x)$ belonging to $R[x]$ can be uniquely written as a finite sum $g(x) = \sum_{i\geqslant 0} g_i(x) f(x)^i$ where for any i, the polynomial $g_i(x)$ belonging to $R[x]$ has degree less than that of $f(x)$. This expansion of $g(x)$ will be referred to as its $f(x)$-expansion.

Let (α, δ) be a (K, v)-minimal pair. The valuation $\tilde{w}_{\alpha,\delta}$ of $\widetilde{K}(x)$ defined on $\widetilde{K}[x]$ by

$$\tilde{w}_{\alpha,\delta}\left(\sum_i c_i (x-\alpha)^i\right) = \min_i\{\tilde{v}(c_i) + i\delta\},\ c_i \in \widetilde{K} \tag{1}$$

will be referred to as the valuation defined by the pair (α, δ). The description of $\tilde{w}_{\alpha,\delta}$ on $K[x]$ is given by the already known theorem stated below (cf. [3,7]).

Theorem 2.A. *Let $\tilde{w}_{\alpha,\delta}$ be the valuation of $\widetilde{K}(x)$ defined by a minimal pair (α, δ) and $w_{\alpha,\delta}$ be the valuation of $K(x)$ obtained by restricting $\tilde{w}_{\alpha,\delta}$. Let $f(x)$ be the minimal polynomial of α over K. Then for any polynomial $g(x)$ in $K[x]$ with $f(x)$-expansion $\sum_{i\geqslant 0} g_i(x) f(x)^i$, one has $w_{\alpha,\delta}(g(x)) = \min_i\{\tilde{v}(g_i(\alpha)) + i w_{\alpha,\delta}(f(x))\}$.*

With the above notations, we prove

Lemma 2.1. *Let (θ, α) be a (K, v)-distinguished pair and $\tilde{w}_{\alpha,\delta}$ be the valuation of $\widetilde{K}(x)$ corresponding to the minimal pair (α, δ) with $\delta = \tilde{v}(\theta - \alpha)$. Let $f(x)$, $g(x)$ be the minimal polynomials over K of α, θ respectively. Then $\tilde{w}_{\alpha,\delta}(g(x)) = \tilde{v}(g(\alpha))$ and $\tilde{w}_{\alpha,\delta}(f(x)) = \tilde{v}(f(\theta))$.*

Proof. Let $\theta^{(i)}$ be any K-conjugate of θ. There exists an automorphism σ of $\widetilde{K}/K$ such that $\sigma(\theta) = \theta^{(i)}$. Since (K, v) is henselian, $\tilde{v} \circ \sigma = \tilde{v}$; so

$$\tilde{v}(\theta^{(i)} - \alpha) = \tilde{v} \circ \sigma(\theta - \sigma^{-1}(\alpha)) = \tilde{v}(\theta - \sigma^{-1}(\alpha)) \leqslant \tilde{v}(\theta - \alpha);$$

consequently by (1), we have, $\tilde{w}_{\alpha,\delta}(x - \theta^{(i)}) = \min\{\delta, \tilde{v}(\alpha - \theta^{(i)})\} = \tilde{v}(\alpha - \theta^{(i)})$. Summing over i, we obtain the first equality. The second equality can be similarly verified. $\square$

Lemma 2.2. *Let (θ, α) and (θ, θ_1) be two (K, v)-distinguished pairs and $g(x)$ be the minimal polynomial of θ over K. Then $\tilde{v}(g(\alpha)) = \tilde{v}(g(\theta_1))$.*

Proof. Denote $\delta_K(\theta) = \tilde{v}(\theta - \alpha) = \tilde{v}(\theta - \theta_1)$ by δ. In view of Lemma 2.1, we have

$$\tilde{v}(g(\alpha)) = \tilde{w}_{\alpha,\delta}(g(x)), \qquad \tilde{v}(g(\theta_1)) = \tilde{w}_{\theta_1,\delta}(g(x)). \tag{2}$$

Keeping in mind that $\tilde{v}(\alpha - \theta_1) \geqslant \delta$, it can be easily checked that the valuations $\tilde{w}_{\alpha,\delta}$ and $\tilde{w}_{\theta_1,\delta}$ are the same. Therefore the lemma follows from (2). $\square$

Lemma 2.3. *Let $g(x)$ and $h(x)$ be two monic irreducible polynomials over a henselian valued field (K, v) of degrees n, m respectively. Let θ be a root of $g(x)$ and γ be a root of $h(x)$. Then $\tilde{v}(g(\gamma)) = \frac{n}{m}\tilde{v}(h(\theta))$.*

Proof. Write $g(x) = \prod_{j=1}^{n}(x - \theta^{(j)})$, $h(x) = \prod_{i=1}^{m}(x - \gamma^{(i)})$. Since $g(x)$, $h(x)$ are irreducible over the henselian valued field (K, v), we have

$$\tilde{v}(g(\gamma^{(i)})) = \tilde{v}(g(\gamma)), \qquad \tilde{v}(h(\theta^{(j)})) = \tilde{v}(h(\theta)), \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Using the equality $\prod_{i=1}^{m} g(\gamma^{(i)}) = \pm \prod_{j=1}^{n} h(\theta^{(j)})$, it follows that $m\tilde{v}(g(\gamma)) = n\tilde{v}(h(\theta))$. $\square$

Lemma 2.4. *Let (θ, θ_1) and (θ_1, θ_2) be two (K, v)-distinguished pairs. Let $f_i(x)$ denote the minimal polynomial of θ_i over K. Then $\tilde{v}(f_1(\theta)) > \frac{\deg f_1}{\deg f_2}\tilde{v}(f_2(\theta_1))$.*

Proof. Set $\delta_1 = \tilde{v}(\theta - \theta_1)$ and $\delta_2 = \tilde{v}(\theta_1 - \theta_2)$. Since $\deg \theta_2 < \deg \theta_1$, it follows from the definition of a distinguished pair that $\tilde{v}(\theta - \theta_2) < \delta_1$; consequently

$$\delta_2 = \tilde{v}(\theta_1 - \theta_2) = \min\{\tilde{v}(\theta_1 - \theta), \tilde{v}(\theta - \theta_2)\} = \tilde{v}(\theta - \theta_2) < \delta_1. \tag{3}$$

If θ_i' runs over all roots of $f_i(x)$ (counted with multiplicities, if any), then $\tilde{v}(f_1(\theta)) = \sum_{\theta_1'} \tilde{v}(\theta - \theta_1')$. Since $\tilde{v}(\theta - \theta_1') \leqslant \delta_1$, it is clear that $\tilde{v}(f_1(\theta)) = \sum_{\theta_1'} \min\{\tilde{v}(\theta - \theta_1'), \delta_1\}$. Keeping in view that $\tilde{v}(\theta - \theta_1) = \delta_1$, it can be easily seen that for any K-conjugate θ_1' of θ_1, we have

$$\min\{\tilde{v}(\theta - \theta_1'), \delta_1\} = \min\{\tilde{v}(\theta_1 - \theta_1'), \delta_1\};$$

consequently

$$\tilde{v}(f_1(\theta)) = \sum_{\theta_1'} \min\{\tilde{v}(\theta_1 - \theta_1'), \delta_1\}.$$

As pointed out in (3), $\delta_1 > \delta_2$. Therefore the last equation shows that

$$\tilde{v}(f_1(\theta)) > \sum_{\theta_1'} \min\{\tilde{v}(\theta_1 - \theta_1'), \delta_2\}. \tag{4}$$

Using the fact that $\tilde{v}(\theta_1 - \theta_2) = \delta_2$, it can be easily verified that

$$\min\{\tilde{v}(\theta_1 - \theta_1'), \delta_2\} = \min\{\tilde{v}(\theta_1' - \theta_2), \delta_2\}. \tag{5}$$

Note that for each K-conjugate θ_1' of θ_1, $\tilde{v}(\theta_1' - \theta_2) \leqslant \delta_K(\theta_1) = \delta_2$. Therefore using (5), we can write (4) as

$$\tilde{v}(f_1(\theta)) > \sum_{\theta_1'} \tilde{v}(\theta_1' - \theta_2) = \tilde{v}(f_1(\theta_2)).$$

In view of Lemma 2.3, $\tilde{v}(f_1(\theta_2)) = \frac{\deg f_1}{\deg f_2}\tilde{v}(f_2(\theta_1))$ and hence the above inequality proves the lemma. $\square$

Notations. For a finite extension L of K contained in $\widetilde{K}$, $\overline{L}$, $G(L)$ will denote respectively the residue field and the value group of the valuation v_L of L obtained by restricting $\tilde{v}$. $def(L/K)$ will stand for the defect of the valued field extension $(L, v_L)/(K, v)$, i.e., $def(L/K) = [L : K]/ef$ where e, f are the index of ramification and residual degree of v_L/v.

The following already known result will be used in what follows. Its proof is omitted (cf. [8]).

Theorem 2.B. *Let (K, v), $(\widetilde{K}, \tilde{v})$ be as above and α, β be elements of $\widetilde{K}$ such that $\tilde{v}(\alpha - \beta) > \tilde{v}(\alpha - \gamma)$ for every γ in $\widetilde{K}$ with $\deg \gamma < \deg \alpha$. Then $G(K(\alpha)) \subseteq G(K(\beta))$, $\overline{K(\alpha)} \subseteq \overline{K(\beta)}$ and $def(K(\alpha)/K)$ divides $def(K(\beta)/K)$.*

The above theorem immediately yields the following corollary.

Corollary 2.C. *If (θ, α) is a (K, v)-distinguished pair and $K(\theta)/K$ is a tamely ramified extension, then so is $K(\alpha)/K$.*

3. Proof of Theorem 1.1

In what follows, we shall write $\tilde{v}(a)$ as $v(a)$ for a belonging to $\widetilde{K}$. Let $\theta = \theta_0, \theta_1, \ldots, \theta_s$ be a complete distinguished chain for θ; as $K(\theta)/K$ is a defectless extension such a chain exists in view of [2, Theorem 1.2]. Let $f_i(x)$ denote the minimal polynomial of θ_i over K of degree n_i and n stand for the degree of $g(x)$. We shall denote $\delta_K(\theta_{i-1})$ by δ_i. In view of (3), $\delta_i > \delta_{i+1}$, $1 \leqslant i \leqslant s-1$. Write $g(x) = \prod_{\theta'}(x-\theta')$. Suppose to the contrary that

$$v(\theta' - \beta) \leqslant \delta_K(\theta) = \delta_1 \quad \text{for every } K\text{-conjugate } \theta' \text{ of } \theta. \tag{6}$$

First it will be shown that assumption (6) implies that

$$v(\theta_1 - \beta') < \delta_1 \quad \text{for every } K\text{-conjugate } \beta' \text{ of } \beta. \tag{7}$$

If there exists a K-conjugate β'' of β with $v(\theta_1 - \beta'') \geqslant \delta_1$, then keeping in mind (6) and the fact that $v(\theta' - \theta_1) \leqslant \delta_1$ for any K-conjugate θ' of θ, it can be easily verified that $v(\theta' - \beta'') = v(\theta' - \theta_1)$; consequently summing over θ', we would have $v(g(\beta'')) = v(g(\theta_1))$, i.e., $v(g(\beta)) = v(g(\theta_1)) = v(g(\alpha))$ in view of Lemma 2.2 which is contrary to the hypothesis. Hence (7) holds.

Let $M(x)$ denote the minimal polynomial of β over K of degree m. We now prove that

$$v(M(\theta)) = v(M(\theta_1)). \tag{8}$$

Let β' be any K-conjugate of β. Then it is clear from (7) and the strong triangle law that

$$v(\theta - \beta') = \min\{v(\theta - \theta_1), v(\theta_1 - \beta')\} = v(\theta_1 - \beta') \tag{9}$$

and hence summing over β', (8) is proved. It is immediate from (8) and Lemma 2.3 that

$$v(M(\theta_1)) = v(M(\theta)) = \frac{m}{n} v(g(\beta)). \tag{10}$$

Using the hypothesis $v(g(\beta)) > v(g(\alpha)) = v(g(\theta_1))$ and the equality $v(g(\theta_1)) = \frac{n}{n_1} v(f_1(\theta))$ derived from Lemma 2.3, it follows from (10) that

$$v(M(\theta)) > \frac{m}{n_1} v(f_1(\theta)).$$

By repeated application of Lemma 2.4, the above inequality gives

$$v(M(\theta)) > \frac{m}{n_i} v(f_i(\theta_{i-1})) \quad \text{for } 1 \leqslant i \leqslant s. \tag{11}$$

Let $\tilde{w}_{\theta_i,\delta_i}$ denote the valuation of $\widetilde{K}(x)$ with respect to the minimal pair (θ_i, δ_i) and w_{θ_i,δ_i} its restriction to $K(x)$. Then by the second assertion of Lemma 2.1, we have

$$\tilde{w}_{\theta_i,\delta_i}(f_i(x)) = v(f_i(\theta_{i-1})), \quad 1 \leqslant i \leqslant s. \tag{12}$$

Let $r \geqslant 1$ be the largest integer such that

$$v(\theta - \beta') < \delta_r \quad \text{for every } K\text{-conjugate } \beta' \text{ of } \beta; \tag{13}$$

such an r exists in view of (9) and (7). The desired contradiction will be obtained by showing that (11) does not hold either for $i = r$ or for $i = r+1$. We first show that

$$v(M(\theta)) = w_{\theta_r,\delta_r}(M(x)). \tag{14}$$

Keeping in mind (3), note that $v(\theta - \theta_r) = \min_{1\leqslant i\leqslant r}\{v(\theta_{i-1} - \theta_i)\} = \delta_r$. Therefore in view of (13), for any K-conjugate β' of β, we have

$$v(\theta - \beta') = v(\theta_r - \beta') = \tilde{w}_{\theta_r,\delta_r}(x - \beta').$$

Summing over β', (14) is proved. Further proof is split in two cases.

Case I. n_r divides m. Denote m/n_r by t. Let $M(x) = f_r(x)^t + M_{t-1}(x)f_r(x)^{t-1} + \cdots + M_0(x)$ be the $f_r(x)$-expansion of $M(x)$. It is immediate from (14), Theorem 2.A and (12) that

$$v(M(\theta)) = w_{\theta_r,\delta_r}(M(x)) \leqslant t w_{\theta_r,\delta_r}(f_r(x)) = \frac{m}{n_r} v(f_r(\theta_{r-1}))$$

which contradicts (11) for $i = r$. Thus the theorem is proved in this case.

Case II. n_r does not divide m. So $n_r \geqslant 2$ and consequently by the definition of a complete distinguished chain $s \geqslant r+1$. We first show that n_{r+1} divides m, this is obvious if $s = r+1$, i.e., $n_{r+1} = 1$. When $s \geqslant r+2$, then keeping in mind that r is the largest positive integer satisfying (13), we see that there exists a K-conjugate β'' of β such that

$$v(\theta - \beta'') \geqslant \delta_{r+1} > \delta_{r+2}.$$

Since $v(\theta - \theta_{r+1}) = \min_{0\leqslant i\leqslant r}\{v(\theta_i - \theta_{i+1})\} = \delta_{r+1} > \delta_{r+2}$, the above inequality gives $v(\theta_{r+1} - \beta'') > \delta_{r+2} = \delta_K(\theta_{r+1})$. It now follows from Theorem 2.B that n_{r+1} divides $m = deg\, \beta''$.

Arguing exactly as in the proof of *Case* I, we see that $w_{\theta_{r+1},\delta_{r+1}}(M(x)) \leqslant \frac{m}{n_{r+1}} v(f_{r+1}(\theta_r))$ which will contradict (11) for $i = r+1$ once we show that

$$w_{\theta_{r+1},\delta_{r+1}}(M(x)) = v(M(\theta)). \tag{15}$$

To verify (15), observe that for any K-conjugate β' of β, we have

$$v(\theta_r - \beta') \leqslant \delta_{r+1}, \tag{16}$$

because otherwise by Theorem 2.B, n_r divides m which is not the case under consideration. Using (16) and the fact that $v(\theta - \theta_r) = \delta_r > \delta_{r+1}$, it can be easily seen that

$$v(\theta - \beta') = v(\theta_r - \beta') = \min\{v(\theta_{r+1} - \beta'), \delta_{r+1}\} = \tilde{w}_{\theta_{r+1},\delta_{r+1}}(x - \beta').$$

On summing over β', (15) follows and hence the desired result.

Note that $\lambda_g = v(g(\alpha))$ is the smallest constant satisfying the property that whenever β belonging to $\widetilde{K}$ is such that $v(g(\beta)) > \lambda_g$, then there exists a K-conjugate θ' of θ with $v(\theta' - \beta) > \delta_K(\theta)$ because on taking $\beta = \alpha$, we have $v(g(\beta)) = \lambda_g$, but there does not exist any K-conjugate θ' of θ for which $v(\theta' - \alpha) > \delta_K(\theta)$. $\square$

Proof of Corollaries 1.2, 1.3. By Theorem 1.1, there exists a K-conjugate θ' of θ such that

$$v(\theta' - \beta) > \delta_K(\theta). \tag{17}$$

Since $\delta_K(\theta') = \delta_K(\theta)$, it follows from (17) and Theorem 2.B that

$$G(K(\theta')) \subseteq G(K(\beta)), \overline{K(\theta')} \subseteq \overline{K(\beta)}, \mathit{def}(K(\theta')/K) \text{ divides } \mathit{def}(K(\beta)/K. \tag{18}$$

Since (K, v) is henselian, $G(K(\theta')) = G(K(\theta))$, $\overline{K(\theta')} \simeq \overline{K(\theta)}$ and $\mathit{def}(K(\theta')/K) = \mathit{def}(K(\theta)/K)$. So Corollary 1.2 follows immediately from (18). For proving Corollary 1.3, it is given that $K(\theta)/K$ is a tamely ramified extension and hence separable. Therefore Krasner's constant $\omega_K(\theta)$ defined by

$$\omega_K(\theta) = \max\{v(\theta - \theta') | \, \theta' \neq \theta \text{ runs over all } K\text{-conjugates of } \theta\}.$$

must be equal to $\delta_K(\theta)$ in view of [9, Lemma 2.2]. It now follows from (17) and Krasner's Lemma [6, Theorem 4.1.7] that $K(\theta') \subseteq K(\beta)$. $\square$

4. Proof of Theorem 1.4, Corollary 1.5

For an element ξ in the valuation ring of $\tilde{v}$, $\bar{\xi}$ will denote its $\tilde{v}$-residue, i.e., the image of ξ under the canonical homomorphism from the valuation ring of $\tilde{v}$ onto its residue field.

Lemma 4.1. *Let (θ, α) be a (K, v)-distinguished pair and β be an element of $\widetilde{K}$.*

(i) *If $v(\beta - \theta) > \delta_K(\theta)$, then for any polynomial $F(x)$ belonging to $K[x]$ of degree less than deg θ, we have $v(F(\theta)) = v(F(\beta))$.*

(ii) *If $A(x) \neq 0$ belonging to $K[x]$ has degree less than deg α, then $\overline{\left(\frac{A(\theta)}{A(\alpha)}\right)} = \bar{1}$.*

Proof. Write $F(x) = c\prod_i (x - \gamma_i)$. Since $\mathit{deg}\ \gamma_i \leqslant \mathit{deg}\ F(x) < \mathit{deg}\ \theta$, it follows that $v(\theta - \gamma_i) \leqslant \delta_K(\theta)$. Keeping in mind that $v(\theta - \beta) > \delta_K(\theta)$, by the strong triangle law, we have

$$v(\beta - \gamma_i) = \min\{v(\beta - \theta), v(\theta - \gamma_i)\} = v(\theta - \gamma_i).$$

Summing over i, we see that $v(F(\beta)) = v(F(\theta))$.

Write $A(x) = a\prod(x - \beta_i)$. Since $\mathit{deg}\ \beta_i < \mathit{deg}\ \alpha$, $v(\theta - \beta_i) < \delta_K(\theta)$ and hence $v(\alpha - \beta_i) = v(\theta - \beta_i) < \delta_K(\theta) = v(\theta - \alpha)$. So $\overline{\left(\frac{A(\theta)}{A(\alpha)}\right)} = \prod \overline{\left(1 + \frac{\theta - \alpha}{\alpha - \beta_i}\right)} = \bar{1}$. $\square$

Lemma 4.2. (i) *Let θ be an element of $\widetilde{K} \setminus K$. For any polynomial $F(x)$ in $K[x]$ of degree less than deg θ, one has $v(F'(\theta)) \geqslant v(F(\theta)) - \delta_K(\theta)$.*

(ii) *Let (θ, α) be a distinguished pair, $K(\theta)/K$ be a tamely ramified extension and $f(x)$ be the minimal polynomial of α over K. Then $v(f'(\theta)) = v(f(\theta)) - \delta_K(\theta)$.*

Proof. Write $F(x) = c\prod_i (x - \gamma_i)$. Since $v(\theta - \gamma_i) \leqslant \delta_K(\theta)$, assertion (i) follows immediately from the equality $F'(\theta) = \sum \frac{F(\theta)}{\theta - \gamma_i}$ by virtue of the triangle law.

Note that assertion (ii) of the lemma is obvious when α belongs to K, in which case $v(f(\theta)) = v(\theta - \alpha) = \delta_K(\theta)$ and $f'(\theta) = 1$. So assume that α is not in K. In view of Corollary 2.C, $K(\alpha)/K$ is tamely ramified and hence separable. Therefore Krasner's constant $\omega_K(\alpha) = \delta_K(\alpha)$ by [9, Lemma 2.2]. Since $K(\alpha)/K$ is defectless, α has a complete distinguished chain; in particular there exists α_1 in $\widetilde{K}$ such that (α, α_1) is a distinguished pair. So $\delta_K(\alpha) = v(\alpha - \alpha_1)$. Using (3), we see that

$$\omega_K(\alpha) = \delta_K(\alpha) = v(\alpha - \alpha_1) < \delta_K(\theta) = v(\theta - \alpha). \tag{19}$$

Since $K(\alpha)/K$ is a separable extension, $f(x) = \prod_i (x - \alpha^{(i)})$ has distinct roots. Set $\alpha^{(1)} = \alpha$. We now verify that for $i > 1$,

$$v(\theta - \alpha^{(i)}) < \delta_K(\theta), \tag{20}$$

because the inequality $v(\theta-\alpha^{(i)}) \geqslant \delta_K(\theta)$ would imply

$$v(\alpha^{(i)}-\alpha) \geqslant \min\{v(\alpha^{(i)}-\theta), v(\theta-\alpha)\} = v(\theta-\alpha) = \delta_K(\theta),$$

which in turn shows that $\omega_K(\alpha) \geqslant \delta_K(\theta)$ contradicting (19). Using the equality $f'(\theta) = \sum_{i\geqslant 1}\frac{f(\theta)}{\theta-\alpha^{(i)}}$, the desired assertion follows from (20) and the strong triangle law. $\square$

Proof of Theorem 1.4. Let $f(x)$ denote the minimal polynomial of α over K and k the smallest positive integer such that $kv(f(\theta)) \in G(K(\alpha))$, say $kv(f(\theta)) = v(h(\alpha))$, $h(x) \in K[x]$, $deg\, h(x) < deg\,\alpha$. In view of Theorem 2.B, $[G(K(\theta)) : G(K(\alpha))]$ divides $deg\,\theta/deg\,\alpha$. Since k divides $[G(K(\theta)) : G(K(\alpha))]$ by Lagrange's Theorem, it must divide ($deg\,\theta/deg\,\alpha) = d$ (say). We shall denote d/k by l. Let $g(x) = \sum_{i=0}^{d} g_i(x)f(x)^i$ be the $f(x)$-expansion of $g(x)$. With notations as in Lemma 2.1, we have

$$\tilde{w}_{\alpha,\delta}(g(x)) = \tilde{v}(g(\alpha)), \qquad \tilde{w}_{\alpha,\delta}(f(x)) = \tilde{v}(f(\theta)) = \lambda(\text{say}).$$

So by Theorem 2.A,

$$v(g_i(\alpha)) + i\lambda \geqslant v(g(\alpha)) \tag{21}$$

with strict inequality if i is not divisible by k. By Theorem 1.1 and the fact that (K, v) is henselian, there exists a K-conjugate β' of β such that $v(\theta-\beta') > \delta_K(\theta) = \delta$ (say). Replacing β' by β, we may assume without loss of generality that

$$v(\theta-\beta) > \delta. \tag{22}$$

Denote the sums $\sum_{k|i} g_i(x)f(x)^i$, $\sum_{k\nmid i} g_i(x)f(x)^i$ by $H(x)$ and $H_1(x)$ respectively, so that $g(x) = H(x) + H_1(x)$. We first show that

$$v(H_1'(\beta)) > v(g(\alpha)) - \delta_K(\theta); \tag{23}$$

this will be accomplished by showing that for each i, one has

$$v(g_i'(\beta)) + i\lambda > v(g(\alpha)) - \delta_K(\theta), \tag{24}$$

and for i not divisible by k, we have

$$v(g_i(\beta)) + (i-1)\lambda + v(f'(\beta)) > v(g(\alpha)) - \delta_K(\theta). \tag{25}$$

Clearly (24) needs to be verified only when $deg\,\alpha > 1$, otherwise each $g_i(x)$ would be constant. Note that by (22) and Lemma 4.1, $v(g_i'(\beta)) = v(g_i'(\theta)) = v(g_i'(\alpha))$. Keeping in mind Lemma 4.2(i) and (19), we see that $v(g_i'(\alpha)) \geqslant v(g_i(\alpha)) - \delta_K(\alpha) > v(g_i(\alpha)) - \delta_K(\theta)$; consequently in view of (21), we have

$$v(g_i'(\beta)) + i\lambda > v(g_i(\alpha)) + i\lambda - \delta_K(\theta) \geqslant v(g(\alpha)) - \delta_K(\theta)$$

which proves (24). Note that by virtue of (22), Lemmas 4.1, 4.2(ii), we have

$$v(f'(\beta)) = v(f'(\theta)) = v(f(\theta)) - \delta_K(\theta) = \lambda - \delta_K(\theta). \tag{26}$$

Using (26) and arguing as for the proof of (24), one can verify (25). Thus (23) is proved. Therefore the theorem is proved once it is shown that

$$v(H'(\beta)) = v(g(\alpha)) - \delta_K(\theta). \tag{27}$$

Taking the derivative of $H(x) = \sum_{k|i} g_i(x)f(x)^i$, we have

$$\begin{aligned} H'(x) &= g_0'(x) + g_k'(x)f(x)^k + \cdots + g_{k(l-1)}'(x)f(x)^{k(l-1)} \\ &\quad + kf(x)^{k-1}f'(x)[lf(x)^{k(l-1)} + (l-1)f(x)^{k(l-2)}g_{k(l-1)}(x) + \cdots + g_k(x)]. \end{aligned}$$

It is clear from (24) that

$$v\left(\sum_{j=0}^{l-1} g_{jk}'(\beta)f(\beta)^{jk}\right) \geqslant \min_j\{v(g_{jk}'(\beta)) + jk\lambda\} > v(g(\alpha)) - \delta_K(\theta).$$

Therefore keeping in mind (26), the desired equality (27) is proved once we show that

$$v(k) + v\left(lf(\beta)^{k(l-1)} + (l-1)g_{k(l-1)}(\beta)f(\beta)^{k(l-2)} + \cdots + g_k(\beta)\right) = v(g(\alpha)) - k\lambda.$$

Recall that $v(g(\alpha)) = \frac{deg\,g}{deg\,f}v(f(\theta)) = kl\lambda$ by virtue of Lemma 2.3; also $v(k) = 0$ as $K(\theta)/K$ is tamely ramified. So in view of (22) and Lemma 4.1, for verifying the above equality, it is enough to show that

$$v\left(lf(\theta)^{k(l-1)} + (l-1)g_{k(l-1)}(\theta)f(\theta)^{k(l-2)} + \cdots + g_k(\theta)\right) = kl\lambda - k\lambda. \tag{28}$$

Define a polynomial $G(Y)$ in an indeterminate Y over $\overline{K(\alpha)}$ by

$$G(Y) = Y^l + \overline{\left(\frac{g_{k(l-1)}(\alpha)}{h(\alpha)}\right)}Y^{l-1} + \cdots + \overline{\left(\frac{g_0(\alpha)}{h(\alpha)^l}\right)}.$$

Set $\xi = \frac{f(\theta)^k}{h(\alpha)}$. In view of Lemma 2.1 and Theorem 2.A, for each i, we have $v(g_i(\theta)f(\theta)^i) \geqslant v(g(\alpha)) = kl\lambda = v(h(\alpha)^l)$ with the inequality being strict when k does not divide i. Consequently on taking the image of the equation

$$0 = \frac{g(\theta)}{h(\alpha)^l} = \frac{f(\theta)^d}{h(\alpha)^l} + \sum_i \frac{g_i(\theta)f(\theta)^i}{h(\alpha)^l}$$

in the residue field and using $\overline{g_i(\theta)/g_i(\alpha)} = \bar{1}$ obtained from Lemma 4.1(ii) for non-zero $g_i(x)$, we see that

$$\bar{\xi}^l + \sum_{r<l} \overline{\left(\frac{g_{kr}(\alpha)}{h(\alpha)^{l-r}}\right)} \bar{\xi}^r = \bar{0}. \tag{29}$$

Since $\bar{\xi}$ is algebraic of degree l over $\overline{K(\alpha)}$ by [1, Section 3], it follows from (29) that $G(Y)$ is the minimal polynomial of $\bar{\xi}$ over $\overline{K(\alpha)}$. As $\overline{K(\theta)}/\overline{K}$ is a separable extension, $\bar{\xi}$ is a simple root of $G(Y)$, i.e., $G'(\bar{\xi}) \neq \bar{0}$. Thus we conclude that $v\left(l\left(\frac{f(\theta)^k}{h(\alpha)}\right)^{l-1} + (l-1)\frac{g_{k(l-1)}(\theta)}{h(\alpha)}\left(\frac{f(\theta)^k}{h(\alpha)}\right)^{l-2} + \cdots + \frac{g_k(\theta)}{h(\alpha)^{l-1}}\right) = 0$, which immediately gives (28). This completes the proof of the theorem. $\square$

Proof of Corollary 1.5. Since $K(\theta)/K$ is tamely ramified, we have $\delta_K(\theta) = \omega_K(\theta) = \tilde{v}(\theta - \theta')$ for some K-conjugate θ' of θ (cf. [9, Lemma 2.2]) and hence $v(\theta' - \alpha) = v(\theta - \alpha) = \delta_K(\theta)$. Therefore keeping in mind that $g(x)$ has coefficients in the valuation ring of v, we have $v(g(\alpha)) \geqslant v(\alpha - \theta) + v(\alpha - \theta') = 2\delta_K(\theta)$. The corollary now follows immediately from Theorem 1.4. $\square$

Acknowledgements

The author is extremely grateful to Professor Ron Brown, University of Hawaii for sending her a preprint of his paper [5]. The financial assistance by National Board for Higher Mathematics, Mumbai is gratefully acknowledged.

References

[1] K. Aghigh, S.K. Khanduja, On the main invariant of elements algebraic over a henselian valued field, Proc. Edinb. Math. Soc. 45 (2002) 219–227.
[2] K. Aghigh, S.K. Khanduja, On chains associated with elements algebraic over a henselian valued field, Algebra Colloq. 12 (4) (2005) 607–616.
[3] V. Alexandru, N. Popescu, A. Zaharescu, A theorem of characterization of residual transcendental extensions of a valuation, J. Math. Kyoto Univ. 28 (4) (1988) 579–592.
[4] R. Brown, Valuations, primes and irreducibility in polynomial rings and rational function fields, Trans. Amer. Math. Soc. 174 (1972) 451–488.
[5] R. Brown, Roots of irreducible polynomials in tame henselian extension fields, Comm. Algebra 37 (7) (2009) 2169–2183.
[6] A.J. Engler, A. Prestel, Valued Fields, Springer-Verlag, Berlin, 2005.
[7] S.K. Khanduja, On valuations of $K(x)$, Proc. Edinb. Math. Soc. 35 (1992) 419–426.
[8] S.K. Khanduja, J. Saha, A generalized fundamental principle, Mathematika 46 (1999) 83–92.
[9] S.K. Khanduja, On Krasner's constant, J. Algebra 213 (1999) 225–230.